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Wave diffraction in a two-fluid system

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Wave diffraction due to a step change in bottom topography is considered for the case of two superimposed fluids of different, but constant, densities. The interface lies below the upper surface of the step. Shallow water theory is shown to be applicable only if the ratio of a non-dimensional frequency parameter to the departure of the density ratio from unity is sufficiently small. An approximate solution of the full equations, obtained by a method applied by Miles (1967) to surface wave diffraction, yields results limited only by the condition that the frequency parameter be small.

1. Introduction

It has recently been suggested (Radok, Munk & Isaacs 1967) that the observed peaking of the incoherent energy of the mid-ocean surface displacement at tidal frequencies might be due in part to the transmission of deep-sea internal waves into shallower water as surface waves. An analysis, based upon the shallow water approximation and the two-fluid model shown in figure 1, indicated that such transmission can occur. Rattray (1960) has also applied this type of analysis to the problem of coastal generation of internal tides.

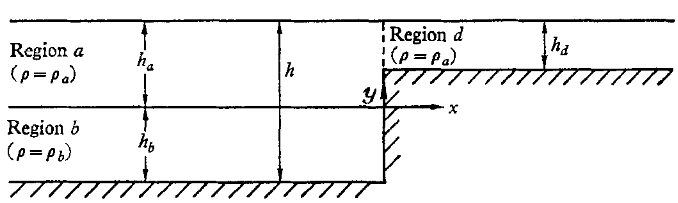

FIGURE 1. Schematic of the two-fluid model.

The amplitudes of the incident surface and internal waves arriving from deep water may be taken as known quantities; the problem is then to solve for the amplitudes of the transmitted surface wave and reflected surface and internal waves. Three conditions must be applied at the step in order to obtain a unique solution. Continuity of surface elevation and continuity of mass flow within each layer would appear to be a reasonable extension of those conditions applied by Lamb (1932, § 176) to the homogeneous case (Lamb's results have been shown by Bartholomeusz (1958) to be the correct asymptotic limits of the exact solutions). On this basis, the analysis yields Lamb's results for the transmitted and reflected surface wave amplitudes, due to an incident surface wave, as the density ratio (ρ_a/ρ_b) is allowed to approach unity. However, the amplitude of the reflected internal wave due to an incident surface wave tends to infinity in this limit and the same result would hold for the transmitted internal wave if the interface were to lie above the step.

These fallacious results (which are presented explicitly in § 4) are a consequence of the conditions applied at the step because only continuity of total mass flow should be applied as (ρ_a/ρ_b) approaches unity. The above conditions place a constraint on the problem such that the internal wave amplitude becomes infinitely large in order to avoid total reflexion of the surface wave. The basic difficulty, however, lies in the breakdown of the shallow water approximation as (ρ_a/ρ_b) approaches unity for a fixed, though small, value of the non-dimensional frequency parameter.

In the present investigation, an approximate solution to the full equations of motion is obtained which is valid for values of (ρ_a/ρ_b) arbitrarily close to unity. The method is identical to that applied by Miles (1967, hereafter denoted by M) to the homogeneous case.

2. Discussion of the eigenvalues

We first discuss the characteristics of the eigenvalues in order to define precisely the limitations of shallow water theory as well as to obtain necessary information concerning the standing edge waves which are excited in the vicinity of the step.

If we define a velocity potential $\tilde{\phi}$ as

$$\mathbf{v} = \mathrm{Re}\{e^{-i\sigma t}\nabla\tilde{\phi}(x,y)\}, \tag{2.1}$$

where

$$\nabla^2\tilde{\phi} = 0 \tag{2.2}$$

within each layer, and if we consider progressive waves of the type

$$\tilde{\phi} = \phi(y)\exp(iKx), \tag{2.3}$$

where K is real, then the exact eigenvalue relation for the region $x < 0$, as given by Lamb (1932, § 231), is

$$\bar{K}_0^2[1-\gamma+\coth\bar{K}H_a\coth\bar{K}H_b]-\bar{K}_0\,\bar{K}[\coth\bar{K}H_a+\coth\bar{K}H_b]+\gamma\bar{K}^2 = 0, \tag{2.4}$$

where

$$\bar{K}_0 = K_0 h = \sigma^2 h/g, \quad \bar{K} = Kh, \tag{2.5}$$

$$H_a = h_a/h, \quad H_b = h_b/h \tag{2.6}$$

and

$$\gamma = 1-(\rho_a/\rho_b). \tag{2.7}$$

The model is meaningful mainly for $\bar{K}_0 \ll 1$ and our later results will be restricted to this case. The eigenvalues then depend on the parameter

$$\lambda^{-1} = \bar{K}_0/\gamma^*, \tag{2.8}$$

where $\gamma^* = \gamma H_a H_b$. We shall consider the depth ratios to be held fixed while $\bar{K}_0$ and γ vary. The subscripts (1, 2) denote, respectively, the surface and internal wave modes. For λ of order unity and $\gamma^* \ll 1$, we find that

$$\bar{K}_1 = \bar{K}_0^{\frac{1}{2}}\{1+\bar{K}_0(\tfrac{1}{6}+\tfrac{1}{2}\lambda)+O(\bar{K}_0^2)\}. \tag{2.9}$$

We can now recover the shallow water surface wave results, namely,

$$\bar{K}_1 = \bar{K}_0^{\frac{1}{2}}\{1 + \tfrac{1}{2}\gamma^* + O(\gamma^*)^2\}, \tag{2.10}$$

by allowing $\bar{K}_0$ to approach zero for fixed γ^*, so $\lambda\bar{K}_0 \gg \bar{K}_0$. On the other hand, if we allow γ^* to approach zero for fixed $\bar{K}_0$, (2.9) gives the first-order correction, for dispersion, to the shallow water result.

For the internal wave, it is appropriate to expand $\bar{K}_2$ as

$$\bar{K}_2 = (\bar{K}_0/\gamma^*)^{\frac{1}{2}}[\hat{K}_{20} + \bar{K}_0\hat{K}_{21} + O(\bar{K}_0^2)]. \tag{2.11}$$

We then find that $\hat{K}_{20}$ is given by the solution of

$$\hat{K}_{20} = H_a H_b \lambda^{-\frac{1}{2}}\{\coth \lambda^{-\frac{1}{2}} H_a \hat{K}_{20} + \coth \lambda^{-\frac{1}{2}} H_b \hat{K}_{20}\}. \tag{2.12}$$

The shallow water result is recovered by allowing λ to tend to infinity, whereupon $\hat{K}_{20}$ tends to unity so that

$$\bar{K}_2 = (\bar{K}_0/\gamma^*)^{\frac{1}{2}}\{1 + O(\gamma^*)\}. \tag{2.13}$$

For shallow water theory to be applicable, $\bar{K}_2$ must be small and so we require $(\bar{K}_0/\gamma^*)^{\frac{1}{2}}$ to be much less than unity. Care should clearly be exercised because a typical value of γ for the ocean is 10^{-3}.

On the other hand, as λ tends to zero, we find that $\hat{K}_{20}$ tends to $2H_a H_b \lambda^{-\frac{1}{2}}$ so that

$$\bar{K}_2 = (2\bar{K}_0/\gamma)\{1 + O(\bar{K}_0)\}. \tag{2.14}$$

The results obtained in this limit might be criticized because the two-fluid model becomes unrealistic for such waves of increasingly short wavelength. Still, the limit does permit us to obtain the correct continuation of the shallow water results and to clarify the difficulties associated with those results. For moderate values of λ, H_a and H_b must first be specified in order to determine $\hat{K}_{20}$ from (2.12).

A similar approach can be used to discuss the characteristics of the standing waves which are excited in the vicinity of the step. If in (2.4) we set $\bar{K} = -i\bar{k}$, where $\bar{k} = kh$, we obtain

$$\bar{K}_0^2[1 - \gamma - \cot \bar{k}H_a \cot \bar{k}H_b] - \bar{K}_0\bar{k}[\cot \bar{k}H_a + \cot \bar{k}H_b] - \gamma\bar{k}^2 = 0. \tag{2.15}$$

We shall consider the special case when $h_b = mh_a$, m being an integer $\geqslant 1$, in which case (2.15) can be rewritten as

$$\bar{K}_0^2 \cos \bar{k} + \bar{K}_0\bar{k} \sin \bar{k} + \gamma(\bar{K}_0^2 + \bar{k}^2) \sin\left(\frac{\bar{k}}{1+m}\right) \sin\left(\frac{m\bar{k}}{1+m}\right) = 0. \tag{2.16}$$

For $\gamma = 0$, the roots are those of

$$\bar{K}_0 = -\bar{k} \tan \bar{k} \tag{2.17}$$

and are infinite in number. Furthermore, it is clear that $\bar{k}$ approaches a multiple of π as $\bar{K}_0$ approaches zero. If n is an integer, we find that

$$\bar{k} = n\pi[1 - (n\pi)^{-2}\bar{K}_0 + O(\bar{K}_0^2)]. \tag{2.18}$$

Now we consider (2.16) and allow $\bar{K}_0$ to approach zero for γ small but fixed. From the last term, we conclude that $\bar{k}$ approaches $(1+m)(n/m)\pi$ and that the case when (n/m) is integral must be considered separately. If we let

$$\bar{k} = (1+m)(n/m)\pi[1 + \bar{K}_0(m/n\pi)^2(1+m)^{-2}\hat{k} + O(\bar{K}_0^2)], \tag{2.19}$$

we find that

$$\hat{k} = -\gamma^{-1} \tag{2.20}$$

if (n/m) is non-integral. When (n/m) is an integer, however, there are two values for $\hat{k}$, namely,

$$\hat{k} = -\{1+\gamma m/(1+m)^2+O(\gamma^2)\} \tag{2.21}$$

and

$$\hat{k} = -\frac{(1+m)^2}{\gamma m}\{1-\gamma m/(1+m)^2+O(\gamma^2)\}. \tag{2.22}$$

These results will be used later when expansion of the transmission coefficients is made.

For the case when γ approaches zero more rapidly than $\overline{K}_0$, say, as an example,

$$\gamma = \hat{\gamma}\overline{K}_0^2, \tag{2.23}$$

if we let

$$\bar{k} = n\pi[1+\overline{K}_0(n\pi)^{-2}\hat{k}+O(\overline{K}_0^2)], \tag{2.24}$$

we find that

$$\hat{k} = -\left\{1+\hat{\gamma}(n\pi)^2(\cos n\pi)^{-1}\sin\left(\frac{n\pi}{1+m}\right)\sin\left(\frac{mn\pi}{1+m}\right)\right\}, \tag{2.25}$$

so that (2.18) is obtained for the special case $\hat{\gamma} = 0$.

3. Analysis for waves of arbitrary wavelength

Referring to figure 1, we fix the origin of the y co-ordinate at the interface. For $x < 0$, the velocity potentials are written as

$$\tilde{\phi}_{a,b} = \sum_{j=1,2}(A_j e^{iK_jx}+\hat{A}_j e^{-iK_jx})\,\Phi_{ja,b}(y)+\sum_q B_q e^{k_qx}\,\phi_{qa,b}(y), \tag{3.1}$$

where either a or b is to be taken as a subscript.

The boundary conditions are

$$(\partial\tilde{\phi}_a/\partial y)-K_0\tilde{\phi}_a = 0 \quad \text{at} \quad y = h_a, \tag{3.2}$$

$$\rho_a\left(\frac{\partial\tilde{\phi}_a}{\partial y}-K_0\tilde{\phi}_a\right) = \rho_b\left(\frac{\partial\tilde{\phi}_b}{\partial y}-K_0\tilde{\phi}_b\right) \quad \text{at} \quad y = 0, \tag{3.3}$$

$$\frac{\partial\tilde{\phi}_a}{\partial y} = \frac{\partial\tilde{\phi}_b}{\partial y} \quad \text{at} \quad y = 0, \tag{3.4}$$

$$\frac{\partial\tilde{\phi}_b}{\partial y} = 0 \quad \text{at} \quad y = -h_b, \tag{3.5}$$

and are satisfied if the K_j and k_q satisfy (2.4) and (2.15), respectively.

Due to the complicated nature of the various functions, they are listed in the appendix. As defined there, they are orthogonal in the sense that

$$\rho_a\int_0^{h_a}\psi_a\Psi_a\,dy+\rho_b\int_{-h_b}^0\psi_b\Psi_b\,dy = \begin{Bmatrix}0, \psi \neq \Psi\\ 1, \psi = \Psi\end{Bmatrix}, \tag{3.6}$$

where ψ or Ψ denotes the eigenfunction associated with any particular progressive or standing wave.

For $x > 0$ only a surface wave is possible, and so

$$\tilde{\phi}_d = \{A_3 e^{iK_3x}+\hat{A}_3 e^{-iK_3x}\}\,\Phi_3(y)+\sum_q D_q e^{-l_qx}\phi_{qd}(y), \tag{3.7}$$

where $\Phi_3(y)$ and $\phi_{qd}(y)$ are given in the appendix by (A 3) and (A 6), respectively, and K_3 and the l_q satisfy

$$K_0 = K_3\tanh K_3h_d = -l_q\tan l_qh_d. \tag{3.8}$$

The functions $\Phi_3(y)$ and $\phi_{qd}(y)$ are orthogonal in the usual sense.

An approximate solution is now obtained by the variational method used by Miles. In order to avoid repetition, only a description of the steps is offered here; the reader is referred to Miles' paper for further details. The Fourier coefficients are first expressed in terms of the unknown horizontal component of velocity at the step, $U(y)$, say, by making use of the orthogonality of the functions. A second imposed condition is that ϕ must be continuous at $x = 0$. By defining the vectors

$$\mathbf{A}_{\mathrm{I}} = \{A_1, A_2, -\hat{A}_3\}, \tag{3.9}$$

$$\mathbf{A}_{\mathrm{II}} = \{\hat{A}_1, \hat{A}_2, -A_3\} \tag{3.10}$$

and

$$\mathbf{\Phi}(y) = \{\Phi_{1a}(y), \Phi_{2a}(y), \Phi_3(y)\}, \tag{3.11}$$

these two conditions can be used to define a scattering matrix $\mathbf{S}$ by the relation

$$\mathbf{K}(\mathbf{A}_{\mathrm{I}} - \mathbf{A}_{\mathrm{II}}) = -i\mathbf{S}(\mathbf{A}_{\mathrm{I}} + \mathbf{A}_{\mathrm{II}}), \tag{3.12}$$

where $\mathbf{K}$ is the diagonal matrix

$$\mathbf{K} = [\delta_{mn} K_m]. \tag{3.13}$$

We can also define from (3.12) a reflexion-transmission matrix $\mathbf{T}$ such that

$$\mathbf{A}_{\mathrm{II}} = \mathbf{T}\mathbf{A}_{\mathrm{I}}, \tag{3.14}$$

where

$$\mathbf{T} = (\mathbf{K} - i\mathbf{S})^{-1}(\mathbf{K} + i\mathbf{S}). \tag{3.15}$$

The unknown $U(y)$ is expressed as

$$U(y) = (\mathbf{A}_{\mathrm{I}} + \mathbf{A}_{\mathrm{II}})\,\mathbf{u}(y), \tag{3.16}$$

as in M (3.8). The elements of $\mathbf{S}$ can be defined in terms of variational integrals, analogous to M (5.2) and defined by (A 13) and (A 14). This fact allows us to approximate the unknown $u_m(y)$. For the surface wave problem, remarkably accurate results were obtained by substituting

$$u_m(y) = \beta_m \Phi_3(y) \tag{3.17}$$

and the same approximation will be used here. We then have, from (A 13, 14) and by use of the orthogonality of $\Phi_3(y)$ and $\phi_{qa}(y)$,

$$\frac{1}{S_{mn}} = \frac{\chi}{N_m N_n}, \tag{3.18}$$

where

$$N_m = \int_{\Delta h}^{h_a} \Phi_m(y)\,\Phi_3(y)\,dy, \tag{3.19}$$

with $\Delta h = h_a - h_d$, and

$$\chi = -\sum_q k_q^{-1}\left\{\int_{\Delta h}^{h_a} \phi_{qa}(y)\,\Phi_3(y)\,dy\right\}^2. \tag{3.20}$$

The full expressions for N_m, $m = 1$ and 2, and χ are given by (A 11) and (A 12); N_3 equals ρ_a^{-1}.

With these results, the transmission and reflexion coefficients can be computed. For instance, the elevation of the free surface for $x < 0$ due to the incident waves is

$$\zeta_{\mathrm{inc}} = iK_0\sigma^{-1} \sum_{j=1,2} \alpha_j |A_j| \exp(iK_j x + i\psi_j), \tag{3.21}$$

where ψ_j is the phase of A_j and

$$\alpha_j = -K_j \Lambda_j^{-1}(K_j \sinh K_j h_a - K_0 \cosh K_j h_a)^{-1}, \tag{3.22}$$

Λ_j being given by (A 7). The elevation of the interface due to the incident waves is

$$\eta_{\text{inc}} = i\sigma^{-1} \sum_{j=1,2} K_j \Lambda_j^{-1} |A_j| \exp(iK_j x + i\psi_j) \tag{3.23}$$

and the elevation of the free surface due to a wave being transmitted into region d is

$$\zeta_{\text{tr}} = iK_0 \sigma^{-1} \alpha_3 A_3 \exp(iK_3 x), \tag{3.24}$$

where

$$\alpha_3 = \Lambda_3^{-1} \cosh K_3 h_d, \tag{3.25}$$

Λ_3 being given by (A 9). Using (3.14) with $\hat{A}_3 = 0$, we can measure the transmission of a surface wave incident from the deeper fluid by the impedance factor Z_1, so that

$$\zeta_{\text{tr}} = Z_1(iK_0 \sigma^{-1} \alpha_1) |A_1| \exp(iK_3 x + i\psi_1) \tag{3.26}$$

and

$$Z_1 = -T_{31}(\alpha_3/\alpha_1). \tag{3.27}$$

The transmission of an incident internal wave is measured by the factor Y_2, where

$$\zeta_{\text{tr}} = Y_2(iK_2/\sigma\Lambda_2) |A_2| \exp(iK_3 x + i\psi_2) \tag{3.28}$$

and

$$Y_2 = -T_{32}(\alpha_3 K_0 \Lambda_2/K_2). \tag{3.29}$$

Thus, ζ_{tr} is related to the elevation of the interface due to the incident internal wave and so Y_2 would seem to have the greatest interest in view of the comments made by Radok *et al.* (1967). However, in view of the discussion given in § 1, it is desirable to find the relation between the elevation of the free surface due to an incident surface wave and the elevation of the interface due to reflexion of that wave as an internal wave. This is measured by the factor $\hat{Y}_1$ where

$$\eta_{\text{refl}} = \hat{Y}_1(iK_0 \sigma^{-1} \alpha_1) |A_1| \exp(-iK_2 x + i\psi_1) \tag{3.30}$$

and

$$\hat{Y}_1 = (K_2/\alpha_1 K_0 \Lambda_2) T_{21}. \tag{3.31}$$

Due to the complexity of the various quantities, we now concentrate on finding the results as $\bar{K}_0$ approaches zero. For the homogeneous case, it is known that the contribution of the standing waves is of order $\bar{K}_0^{\frac{1}{2}}$. We shall first set χ to be zero (plane wave approximation), discuss the results on the basis of § 2 and then finally consider the validity of this approximation by evaluating χ as $\bar{K}_0$ approaches zero.

4. The plane wave approximation

Upon evaluation of the elements of $\mathbf{T}$, we find that

$$T_{31} = -2\rho_a N_1/P, \quad T_{32} = (N_2/N_1) T_{31}, \quad T_{21} = -2\rho_a^2 N_1 N_2 K_3/K_2 P, \tag{4.1}$$

where

$$P = 1 + \rho_a^2\{(N_1^2 K_3/K_1) + (N_2^2 K_3/K_2) + i\chi K_3\}. \tag{4.2}$$

If we now set $\chi = 0$ and evaluate the above expressions for the shallow water limit ($\bar{K}_0 \to 0$ for γ small but fixed), we find that

$$-Z_1 = 2[1 + H_d^{\frac{1}{2}}\{1 + H_b(\gamma H_b/H_a)^{\frac{1}{2}}\} + O(\gamma)]^{-1}, \tag{4.3}$$

where $H_d = h_d/h$,

$$-Y_2 = 2[\gamma H_b(1+H_d^{\frac{1}{2}})^{-1}+O(\gamma^{\frac{3}{2}})] \tag{4.4}$$

and

$$-\hat{Y}_1 = 2(H_b H_d/\gamma H_a)^{\frac{1}{2}}(1+H_d^{\frac{1}{2}})^{-1}. \tag{4.5}$$

Thus, (4.3) yields Lamb's (1932, § 176) result as $\gamma \to 0$, (4.4) indicates the transmission of a surface wave with amplitude γ times the amplitude of an incident internal wave and (4.5) yields, as $\gamma \to 0$, the singular result discussed in the first section of this paper.

On the other hand, if γ tends to zero for $\bar{K}_0$ small but fixed, we find, after use of (2.14), that N_2 vanishes exponentially fast so that $(-Z_1)$ approaches the homogeneous fluid result and Y_2 and $\hat{Y}_1$ tend to

$$Y_2 = -\tfrac{1}{2}(\gamma^2/\bar{K}_0 H_d)(1+H_d^{\frac{1}{2}})^{-1}\exp(-2\bar{K}_0\Delta H/\gamma) \tag{4.6}$$

and

$$\hat{Y}_1 = -2(\bar{K}_0 H_d)^{-\frac{1}{2}}(1+H_d^{\frac{1}{2}})^{-1}\exp(-2\bar{K}_0\Delta H/\gamma), \tag{4.7}$$

where $\Delta H = H_a - H_d$. The exponential behaviour can only be predicted by considering the limits of results obtained on the basis of the full equations of motion.

For intermediate values of $(\bar{K}_0/\gamma)$, we refer to (2.11) and assume that $\bar{K}_2$ is adequately given by the first term in the expansion. Then expressions for Y_2 and $\hat{Y}_1$, accurate to the same degree in $\bar{K}_0$, are given by

$$Y_2 = \frac{-2\bar{K}_0 \sinh \bar{K}_2 H_d}{(1+H_d^{\frac{1}{2}})\,H_a \bar{K}_2^2 \sinh \bar{K}_2 H_a} \tag{4.8}$$

and

$$\hat{Y}_1 = \frac{-4\sinh \bar{K}_2 H_d}{\bar{K}_2\theta(\bar{K}_0 H_d)^{\frac{1}{2}}(1+H_d^{\frac{1}{2}})\sinh \bar{K}_2 H_a}, \tag{4.9}$$

where

$$\theta = (\sinh \bar{K}_2 H_b)^{-2}(H_b + \tfrac{1}{2}\bar{K}_2^{-1}\sinh 2\bar{K}_2 H_b) + \bar{K}_2^{-2}(\sinh \bar{K}_2 H_a)^{-2}(\bar{K}_2^2 H_a + \tfrac{1}{2}\bar{K}_2 \sinh 2\bar{K}_2 H_a). \tag{4.10}$$

These expressions lead to the shallow water results for $(\bar{K}_0/\gamma) \ll 1$ and to (4.6) and (4.7) for $(\bar{K}_0/\gamma) \gg 1$ but $\bar{K}_0 \ll 1$.

5. Discussion of the standing edge wave contribution

We shall now discuss briefly the behaviour of χ, given by (A 12), in order to assess its order of magnitude. For the case $(\bar{K}_0/\gamma) \gg 1$, little needs to be said because, for the homogeneous case, it is known that the effect upon the transmission coefficients is of $O(\bar{K}_0^{\frac{1}{2}})$. Thus, the results (4.6–4.7) still hold, except that the error term is of $O(\bar{K}_0^{\frac{1}{2}})$. On the other hand, the case $(\bar{K}_0/\gamma) \ll 1$ is not settled directly because the distribution of eigenvalues is quite different from the homogeneous case (cf. § 2).

We first consider the case when $\bar{K}_0$ approaches zero but, using the terminology of § 2, (n/m) is not an integer. We note that

$$\sin k_q h_b = \sin\left(\frac{m}{1+m}\right)\bar{k}_q \sim O(\bar{K}_0/\gamma). \tag{5.1}$$

The resulting singularity in $(\Lambda_q^-)^2$ (cf. A 8) will cause χ to be at least of $O(\bar{K}_0^2/\gamma^2)$. We therefore conclude that the major contribution to χ as $\bar{K}_0 \to 0$ must come from those eigenvalues for which (n/m) is an integer say, n', so that

$$\sin k_q h_a = \sin\left(\frac{\bar{k}_q}{1+m}\right) \sim O(\bar{K}_0/\gamma), \tag{5.2}$$

which results in the cancellation of the singularity in $(\Lambda_q^-)^2$. Using the results given by (2.21) and (2.22), it is found, after some algebra, that as $\bar{K}_0$ approaches zero we can write

$$\rho_1^2 K_3 \chi = -2\bar{K}_0^{\frac{1}{2}} H_d^{-\frac{3}{2}} H_a^{-1} \sum_{n'} F_{n'}, \tag{5.3}$$

where

$$F_{n'} = \frac{\sin^2\{(1+m)\,n'\pi H_d\}}{(1+m)^3\,(n'\pi)^3}\left\{\frac{1+m}{m} + \gamma\left(\frac{1+m^2}{m+m^2}\right) + O(\gamma^2)\right\}. \tag{5.4}$$

Therefore the correction to the shallow water results is of $O(\bar{K}_0^{\frac{1}{2}})$ if the series in (5.3) is convergent. But we can say that, if terms of $O(\gamma^2)$ are neglected,

$$\sum_{n'} F_{n'} \leqslant C \sum_{n'} \left(\frac{1}{n'}\right)^3 \leqslant C \int_1^\infty \frac{dx}{x^3} = \frac{C}{2}, \tag{5.5}$$

where

$$C = \frac{1}{\pi^3(1+m)^3}\left\{\frac{1+m}{m} + \gamma\left(\frac{1+m^2}{m+m^2}\right)\right\}. \tag{5.6}$$

The series is therefore convergent.

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Appendix

The following quantities are used in the text.

$$\Phi_{ja}(y) = \frac{K_0 \sinh K_j(h_a - y) - K_j \cosh K_j(h_a - y)}{\Lambda_j(K_j \sinh K_j h_a - K_0 \cosh K_j h_a)}. \tag{A 1}$$

$$\Phi_{jb}(y) = \frac{\cosh K_j(h_b + y)}{\Lambda_j \sinh K_j h_b} \quad (j = 1 \text{ or } 2). \tag{A 2}$$

$$\Phi_3(y) = \Lambda_3^{-1} \cosh\, K_3(y - \Delta h) \quad (\Delta h = h_a - h_d). \tag{A 3}$$

$$\phi_{qa}(y) = \frac{k_q \cos k_q(h_a - y) - K_0 \sin k_q(h_a - y)}{(\Lambda_q^-)\,(k_q \sin k_q h_a + K_0 \cos h_q h_a)}. \tag{A 4}$$

$$\phi_{qb}(y) = -\frac{\cos k_q(h_b + y)}{(\Lambda_q^-) \sin k_q h_b}. \tag{A 5}$$

$$\phi_{qd}(y) = (\Lambda_q^+)^{-1} \cos l_q(y - \Delta h). \tag{A 6}$$

$$\Lambda_j^2 = \rho_b \int_{-h_b}^0 \frac{\cosh^2 K_j(h_b + y)}{\sinh^2 K_j h_b}\, dy + \rho_a \int_0^{h_a} \left[\frac{K_0 \sinh K_j(h_a - y) - K_j \cosh K_j(h_a - y)}{K_j \sinh K_j h_a - K_0 \cosh K_j h_b}\right]^2 dy$$

$$= \frac{\rho_b}{2\sinh^2 K_j h_b}\left\{h_b + \frac{\sinh 2K_j h_b}{2K_j}\right\}$$

$$-\frac{\rho_a}{2}\left\{\frac{(K_0^2 - K_j^2)\, h_a - K_0 + K_0 \cosh 2K_j h_a - \frac{1}{2}K_j^{-1}(K_0^2 + K_j^2) \sinh 2K_j h_a}{(K_j \sinh K_j h_a - K_0 \cosh K_j h_a)^2}\right\}. \tag{A 7}$$

$$(\Lambda_q^-)^2 = \rho_b \int_{-h_b}^{0} \frac{\cos^2 k_q(h_b+y)}{\sin^2 k_q h_b}\, dy + \rho_a \int_0^{h_a} \left[\frac{k_q \cos k_q(h_a-y) - K_0 \sin k_q(h_a-y)}{k_q \sin k_q h_a + K_0 \cos k_q h_a}\right]^2 dy$$

$$= \frac{\rho_b}{2\sin^2 k_q h_b}\left\{h_b + \frac{\sin 2k_q h_b}{2k_q}\right\}$$

$$+ \frac{\rho_a}{2}\left\{\frac{(K_0^2+k_q^2)\, h_a - K_0 + K_0 \cosh 2k_q h_a + \tfrac{1}{2}k_q^{-1}(k_q^2 - K_0^2)\sin 2k_q h_a}{(k_q \sin k_q h_a + K_0 \cos k_q h_a)^2}\right\}. \quad \text{(A 8)}$$

$$\Lambda_3^2 = \tfrac{1}{2}\rho_a\left\{h_d + \frac{1}{2K_3}\sinh 2K_3 h_d\right\} = \rho_a \int_{\Delta h}^{h_a} \cosh^2 K_3(y-\Delta h)\, dy. \quad \text{(A 9)}$$

$$(\Lambda_q^+)^2 = \rho_a \int_{\Delta h}^{h_a} \cos^2 l_q(y-\Delta h)\, dy = \tfrac{1}{2}\rho_a\left\{h_d + \frac{1}{2l_q}\sin 2l_q h_d\right\}. \quad \text{(A 10)}$$

$$N_j\,(j=1,2) = (\Lambda_j\Lambda_3)^{-1}\left(\frac{K_j}{K_j^2-K_3^2}\right)\left\{\frac{K_3 \sinh K_3 h_d - K_j \sinh K_j h_d - K_0 \cosh K_3 h_d + K_0 \cosh K_j h_d}{K_j \sinh K_j h_a - K_0 \cosh K_j h_a}\right\}. \quad \text{(A 11)}$$

$$\chi = -\sum_q \frac{k_q(\Lambda_3\Lambda_q^-)^{-2}}{(k_q^2+K_3^2)^2}\left\{\frac{K_3 \sinh K_3 h_d + k_q \sin k_q h_d - K_0 \cosh K_3 h_d + K_0 \cos k_q h_d}{k_q \sin k_q h_a + K_0 \cos k_q h_a}\right\}^2. \quad \text{(A 12)}$$

$$\frac{1}{S_{mn}} = \frac{\displaystyle\iint_{\Delta h}^{h_a} u_m(\xi)\, G(y,\xi)\, u_n(y)\, dy\, d\xi}{\displaystyle\rho_a \int_{\Delta h}^{h_a} \Phi_m(y)\, u_n(y)\, dy \int_{\Delta h}^{h_a} \Phi_n(\xi)\, u_m(\xi)\, d\xi}. \quad \text{(A 13)}$$

$$G(y,\xi) = -\rho_a \sum_q \{l_q^{-1}\phi_{qd}(y)\,\phi_{qd}(\xi) + k_q^{-1}\phi_{qa}(y)\,\phi_{qa}(\xi)\}. \quad \text{(A 14)}$$

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